Int. J. Heat Mass Transfer. Vol. 18, pp. 1049–1053. Pergamon Press 1975. Printed in Great Britain

CORRELATING EQUATIONS FOR LAMINAR AND TURBULENT FREE CONVECTION FROM A HORIZONTAL CYLINDER

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(*Received* 10 *October* 1974)

Abstract—A simple empirical expression for the mean value of *Nu* over the cylinder for all *Ra* and all *Pr* is developed in terms of the model of Churchill and Usagi. This expression is applicable for uniform heating as well as for uniform wall temperature and for mass transfer and simultaneous heat and mass transfer. Even simpler expressions are obtained for restricted conditions. These expressions improve upon prior graphical and empirical correlations in both accuracy and convenience.

NOMENCLATURE

- a, arbitrary exponent;
- A, dimensionless coefficient;
- b, arbitrary exponent;
- D, diameter of cylinder [m];
- $\mathscr{D}$, diffusivity [m^2/s];
- $f\{Pr\}$, dimensionless function of Pr defined by equation (9);
- g, acceleration due to gravity [m/s^2];
- h, space-mean heat-transfer coefficient [$J/s.m^2.°K$];
- k, thermal conductivity [$J/s.m.°K$];
- k', mass-transfer coefficient [s^{-1}];
- Nu, Nusselt number, hD/k;
- n, arbitrary exponent in Churchill–Usagi model;
- Pr, Prandtl number, ν/α;
- q, heat flux density, $J/s.m^2$;
- Ra, Rayleigh number, $g\beta(T_s - T_b)D^3/\nu\alpha$;
- Ra', Rayleigh number for mass transfer, $g\gamma(\omega_s - \omega_b)D^3/\nu\mathscr{D}$;
- Ra^*, modified Rayleigh number based on heat flux, $g\beta qD^4/k\nu\alpha$;
- Sc, Schmidt number, $\nu/\mathscr{D}$;
- Sh, Sherwood number, $k'D/\mathscr{D}$;
- T, temperature [°K].

Greek symbols

- α, thermal diffusivity [m^2/s];
- β, coefficient for thermal expansion [$°K^{-1}$];
- γ, dimensionless coefficient for expansion due to composition;
- ω, mass fraction of transferring component;
- ν, kinematic viscosity [m^2/s].

Subscripts

- b, bulk;
- s, surface;
- 0, value as $Ra \to 0$.

INTRODUCTION

CHURCHILL and Chu [1] recently presented correlations for the space-mean coefficient for free convection from a vertical plate based on the model of Churchill and Usagi [2]. Similar correlations are developed herein for the space-mean coefficient for free convection from a horizontal cylinder.

Extensive data are available and many correlations have been proposed for the horizontal cylinder. However, the theoretical solutions for the laminar regime are less accurate than for the flat plate owing to the formation of a wake at the rear of the cylinder even at quite low Rayleigh numbers. In the interest of brevity, previous correlations, theoretical solutions and experimental data will not be reviewed or analyzed except insofar as they are directly relevant to the derivations herein. The development closely follows that in [1] and hence will be outlined only.

The derivations are first carried out in terms of heat transfer from an isothermal cylinder and then for a uniform heat flux density at the surface. The results are subsequently generalized for mass transfer and other applications.

Laminar regime

Boundary-layer theory has been utilized to derive numerical solutions for cylinders. Saville and Churchill [3] have shown these solutions to be quite accurate for moderate Rayleigh numbers such that the wake is confined to a small region at the rear of the cylinder. For the asymptotic cases of $Pr \to \infty$ and 0, Lefevre [4] and Saville and Churchill [5], respectively, have derived solutions for the mean Nusselt number which can be expressed as

$$Nu \to 0{\cdot}518Ra^{1/4} \quad \text{as} \quad Pr \to \infty \qquad (1)$$

and

$$Nu \to 0{\cdot}599Ra^{1/4}Pr^{1/4} \quad \text{as} \quad Pr \to 0. \qquad (2)$$

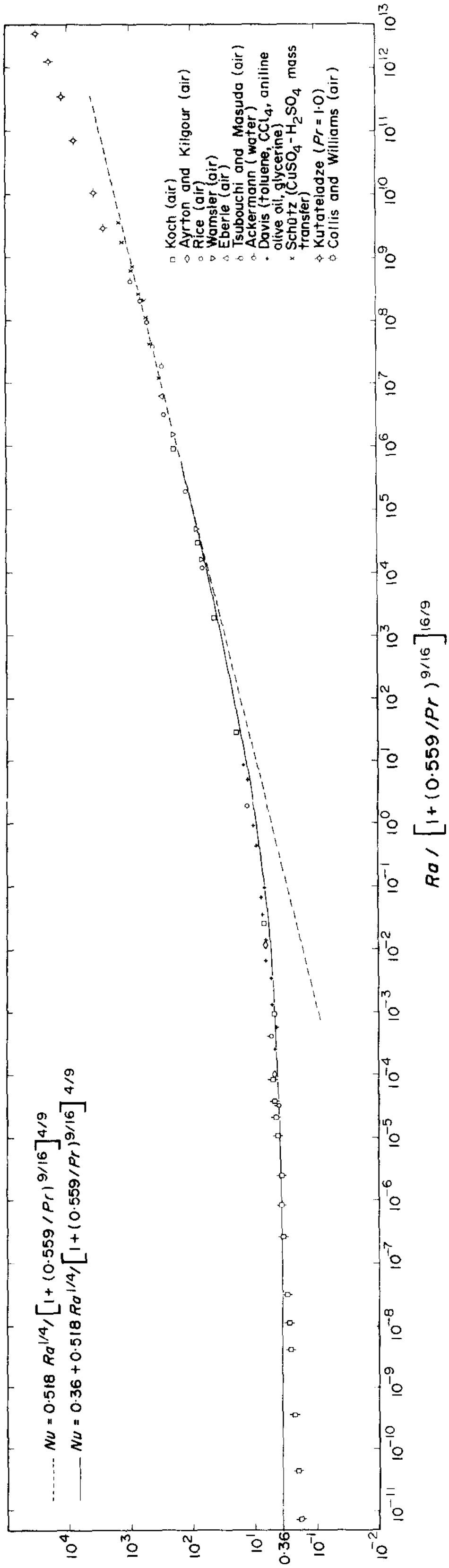

FIG. 1. Comparison of laminar correlating equations with experimental data for heat and mass transfer.

These limiting solutions can be combined in the form suggested by Churchill and Usagi [2] to give the expression:

$$Nu = 0{\cdot}518Ra^{1/4}/[1+(0{\cdot}559/Pr)^{n/4}]^{1/n}. \qquad (3)$$

An exponent n of 9/4 reproduces closely the numerical value of 0·392 computed by Saville and Churchill [3] for $Pr = 0{\cdot}7$ and thus yields the following expression for all Pr:

$$Nu = 0{\cdot}518\left(\frac{Ra}{[1+(0{\cdot}559/Pr)^{9/16}]^{16/9}}\right)^{1/4}. \qquad (4)$$

It may be noted that the exponent of 9/4 is identical to that derived by Churchill and Ozoe [6] for free convection from a vertical flat plate. Equation (4) would be expected to become invalid as Ra increases sufficiently owing to development of the wake and also as Ra decreases sufficiently owing to thickening of the boundary layer relative to the diameter of the cylinder. An accepted solution has not yet been derived for the latter regime. For pure conduction from an infinitely long cylinder $Nu = 0$. However, a significant, finite value is observed experimentally for very small Ra. Tsubouchi and Masuda [7] proposed an empirical, limiting value of 0·36 based on the analysis of several sets of data.

Combining this latter value with equation (4) in the form of the model of Churchill and Usagi results in the following test expression for the entire laminar regime:

$$Nu^n = 0{\cdot}36^n + \left(\frac{0{\cdot}518Ra^{1/4}}{[1+(0{\cdot}559/Pr)^{9/16}]^{4/9}}\right)^n. \qquad (5)$$

Test plots of experimental data in the form suggested by Churchill and Usagi indicate that 1·0 is a suitable value for n, yielding the correlation

$$Nu = 0{\cdot}36 + 0{\cdot}518\left(\frac{Ra}{[1+(0{\cdot}559/Pr)^{9/16}]^{16/9}}\right)^{1/4}. \qquad (6)$$

Equation (6) is seen in Fig. 1 to provide a good representation of representative data [7–15] for all Pr for $10^{-6} < Ra < 10^9$. An exception is the set of experimental values of Collis and Williams [16] for small diameter wires which fall below equation (6) for $Ra < 10^{-6}$ and appear to approach zero.

A correlation for a complete range of Rayleigh number

An asymptotic solution is not available for $Ra \to \infty$ corresponding to a completely turbulent boundary layer and wake. However, experimental data and the previous results for a vertical plate [1] suggest the expression

$$Nu \to A\,Ra^{1/3}f\{Pr\} \quad \text{as} \quad Ra \to \infty \qquad (7)$$

where A is a constant and $f\{Pr\}$ is a function approaching unity for $Pr \to \infty$ and proportionality to $Pr^{1/3}$ for $Pr \to 0$.

The right sides of equations (6) and (7) could be combined to obtain a trial expression for all Ra and Pr. However, the limiting value of 0·36 proves to combine with equation (7) to yield a simpler and equally successful correlation. The resulting test expression is

$$Nu^n = 0{\cdot}36^n + (A\,Ra^{1/3}f\{Pr\})^n. \qquad (8)$$

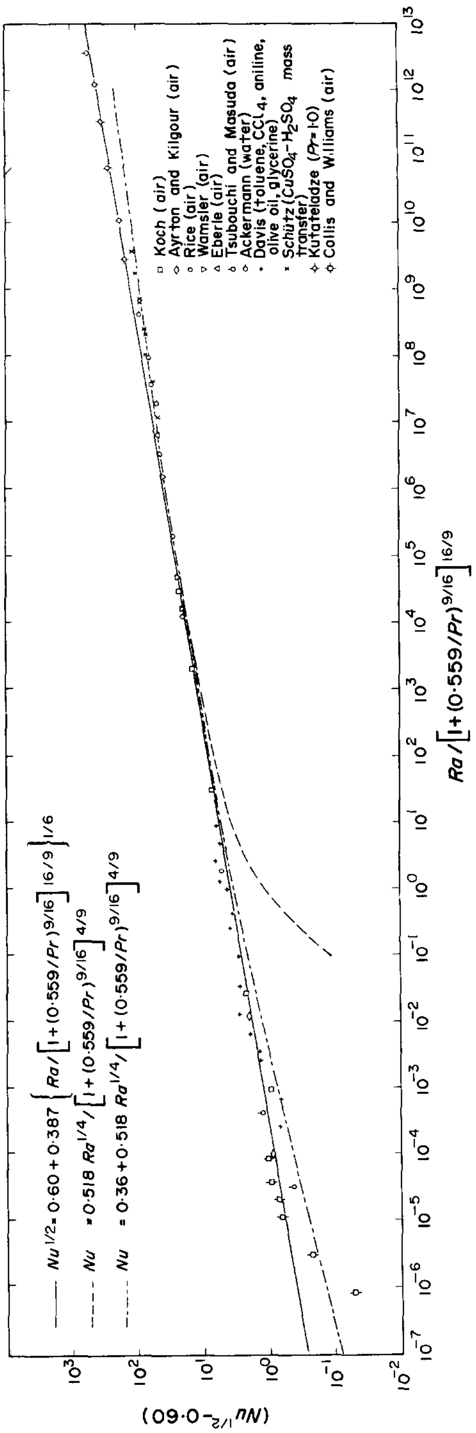

FIG. 2. Comparison of correlating equations with experimental data for heat and mass transfer.

Bosworth [17] suggested a correlation for air equivalent to equation (8) with $n = 1/2$, $Nu_0 = 0{\cdot}397$ instead of 0·36, and $Af\{Pr\} = 0{\cdot}1225$. This value of n proves to be a reasonable choice for all fluids. A general expression for $Af\{Pr\}$ for all fluids is derived below.

Equation (8) provides a dependence on Ra that increases continuously from the zeroth power to the 1/3-power as Ra increases. If equation (8) is to produce the same dependence on Pr as equation (4) in the laminar boundary-layer regime, it is necessary that

$$f\{Pr\} = \left[1+\left(\frac{0{\cdot}559}{Pr}\right)^{9/16}\right]^{(16/9)(-1/3)}. \qquad (9)$$

This function also conforms to the asserted dependence as Pr approaches 0 and ∞ for $Ra \to \infty$.

A straight line with a slope of 1/6 through a plot of $\log(Nu^{1/2}-0{\cdot}6)$ vs $\log Ra/[1+(0{\cdot}599/Pr)^{9/16}]^{16/9}$ for representative data [7–15] in Fig. 2 yields a value of $A^{1/2} = 0{\cdot}387$, or $A = 0{\cdot}15$, and hence the final correlation:

$$Nu^{1/2} = 0{\cdot}60+0{\cdot}387\left(\frac{Ra}{[1+(0{\cdot}559/Pr)^{9/16}]^{16/9}}\right)^{1/6}. \qquad (10)$$

Equations (4) and (6) are also plotted in Fig. 2 for comparison and to indicate their range of applicability.

INTERPRETATION

Values of $f\{Pr\}$, of $f^{3/4}\{Pr\}$ corresponding to the function in equation (4), and of $Pr^{-1/4}f^{3/4}\{Pr\}$ are given in Table 1 for representative fluids. The large deviations for air and water from the limiting dependence for $Pr \to \infty$ indicate why the customary empirical equations of the form of equation (1) have not

Table 1. Deviations from limiting dependence on Pr

Fluid	Pr	$f\{Pr\}$	$f^{3/4}\{Pr\}$	$Pr^{-1/4}f^{3/4}\{Pr\}$
---	0	0	0	1·214
Mercury	~0·02	0·303	0·408	1·085
Air	~0·70	0·688	0·755	0·826
Water	~6·0	0·871	0·901	0·576
Oil	100	0·969	0·977	0·308
---	∞	1·000	1·000	0·000

been completely satisfactory for a variety of fluids with a significant range of Pr. Similar discrepancies are to be expected for correlations for liquid metals based on equation (2). A further variation in the dependence on Pr for all fluids arises from the additive constant in equations (6) and (10). Equations (6) and (10) confirm that correlations of the simple form

$$Nu = A\,Ra^a Pr^b \qquad (11)$$

cannot be successful over an extended range of Ra and Pr and have outlived their usefulness in that regard.

Equation (10) provides a smooth transition from the laminar to the turbulent regime whereas the actual transition is known to occur in a number of discrete steps. The representation by equation (10) is numerically successful because the effect of the transitions is minimized by averaging over the circumference. The impact of the transitions on the local Nusselt number is of course much greater, and a correlation encompassing the full range of Ra would necessarily have a more complicated form than equation (10).

For large temperature differences such that the variation of physical properties is significant, the properties may be evaluated at the average of the bulk and surface temperatures as a first approximation. Wylie [18] suggests a more accurate but a more complicated procedure.

UNIFORM HEAT FLUX

The numerical solution of Wilks [19] for a number of finite values of Pr with uniform heating and a laminar boundary layer has been utilized by Churchill [20] to construct the correlation

$$Nu = 0{\cdot}579\left(\frac{Ra}{[1+(0{\cdot}442/Pr)^{9/16}]^{16/9}}\right)^{1/4}. \quad (12)$$

The values of Nu on which this correlation is based were apparently obtained by averaging local values. Sparrow and Gregg [21] have shown for the vertical plate that use of the temperature difference at half the total distance in the definition of Nu and Ra yields values in much closer agreement with those for uniform wall temperature. For the flat plate the ratio of the coefficients is $2^{11/4}/5^{5/4}$. Arbitrarily applying this factor as an approximation for the horizontal cylinder yields a coefficient of 0·521 rather than 0·579. The resulting expression differs significantly from equation (4) only for very small Pr.

Neither an asymptotic solution nor a limiting value appears to be available for $Ra \to 0$. Assuming the same limiting value and the same exponent of unity as for equation (5) then results in the following expression for the laminar regime:

$$Nu = 0{\cdot}36+0{\cdot}521\left(\frac{Ra}{[1+(0{\cdot}442/Pr)^{9/16}]^{16/9}}\right)^{1/4}. \quad (13)$$

Experimental values to test equation (13) or to construct a correlation encompassing the turbulent regime do not appear to be available. In the absence of such data, equation (10) is recommended as an approximation for uniform heating as well as for uniform surface temperature. Equation (13) is possibly a more accurate expression for the laminar regime for small Pr.

Free convection with uniform heating is often correlated in terms of Ra^* in order to avoid explicit inclusion of the surface temperature. Equations (10), (12) and (13) can be rewritten in terms of Ra^* by replacing $(T_s - T_b)$ in Ra with q/h and resolving for hD/k. However, such a re-expression disguises the important fact that the dependence of Nu on Ra is essentially the same for uniform heating as for uniform wall temperature.

MASS TRANSFER AND SIMULTANEOUS HEAT AND MASS TRANSFER

Equations (6) and (10) with Sh, Sc and Ra' substituted for Nu, Pr and Ra should be applicable for mass transfer as long as the net rate of mass transfer does not produce a significant velocity normal to the surface. Representative mass-transfer data [22] are included in Figs. 1 and 2, and reasonable agreement with the correlating equations can be noted.

On the basis of the results of Saville and Churchill [23] for mass transfer due to a temperature gradient only, $[\gamma(\omega_s - \omega_b)/\beta(T_s - T_b) \to 0$ and $Pr/Sc \to 0]$, Sh can be substituted for Nu and $Ra(Sc/Pr)^{4/3}$ for Ra in equations (6) and (10). Furthermore, Nu and Sh can be calculated from equations (6) and (10) for the special case of $Sc = Pr$ merely by substituting $Ra + Ra'$ for Ra.

CONCLUSIONS

1. Equation (10) based on experimental values for $Ra \to 0$ and ∞ and on equation (4) for the interrelationship between Ra and Pr provides a simple yet good representation for the average Nusselt number for uniform surface temperature for all Pr and Ra, except as noted below, even though it fails to take into account the discrete transitions from the laminar to the turbulent regime. It is probably a good approximation for uniform heating if the temperature difference at 90° is used in the definition of Ra and Nu.

2. Equation (6) based on equation (4) and the limiting value of Tsubouchi and Masuda for small Ra provides a slightly better representation for the experimental values for the average Nusselt number for uniform surface temperature and all Pr for the entire laminar regime ($Ra < 10^9$), except as noted below.

3. The precise experimental values of Collis and Williams for small wires fall below equations (6) and (10) for $Ra < 10^{-6}$ and this is therefore recommended as their lower limit of reliability.

4. Equation (4) based on the asymptotic solutions for $Pr \to 0$ and ∞ provides a good representation for the values of the average Nusselt number computed from laminar boundary-layer theory for uniform surface temperature and all Pr. However, it correlates the experimental data well only for $10^4 < Ra < 10^9$.

5. Equation (13), based on the temperature difference at 90°, provides a possibly better correlation than equations (6) and (10) for uniform heating in the laminar regime for small Pr.

6. Equations (4), (6) and (10) are applicable to mass transfer with Sh, Sc and Ra' substituted for Nu, Pr and Ra, and for simultaneous heat and mass transfer for $Pr = Sc$ with $Ra + Ra'$ substituted for Ra. Other extensions are also possible.

7. The principle uncertainty in the correlations arises from the uncertainty in the limiting solutions and experimental values for $Ra \to 0$ and ∞, and for uniform heating.

8. Simple power-law expressions in the form of equation (11) are fundamentally inadequate for a wide range of either Pr or Ra and should no longer be used for that purpose.

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EQUATIONS DE CORRELATION EN CONVECTION LIBRE LAMINAIRE ET TURBULENTE AUTOUR D'UN CYLINDRE HORIZONTAL

Résumé—Une expression empirique simple de la valeur moyenne de *Nu* autour d'un cylindre valable pour tout *Ra* et tout *Pr* est obtenue à partir du modèle de Churchill et Usagi. Cette expression est applicable aussi bien dans le cas d'un chauffage à flux constant que dans celui d'un chauffage à température de paroi constante ainsi que pour le transfert de masse et le transfert simultané de chaleur et de masse. Des expressions encore plus simples sont obtenues dans des conditions plus restrictives. Ces expressions apportent des améliorations aux corrélations graphiques et empiriques antérieures à la fois en précision et en commodité.

KORRELATIONSGLEICHUNGEN FÜR LAMINARE UND TURBULENTE FREIE KONVEKTION AN EINEM HORIZONTALEN ZYLINDER

Zusammenfassung—Es wurde ein einfacher empirischer Ausdruck für den Mittelwert von *Nu* über den Zylinder für alle *Ra* und alle *Pr* mit Termen des Modells von Churchill und Usagi entwickelt. Dieser Ausdruck kann sowohl verwendet werden für konstante Wärmezufuhr als auch für konstante Wandtemperatur und für Stoffübergang und gleichzeitigen Wärme- und Stoffübergang. Man erhält noch einfachere Ausdrücke für einschränkende Bedingungen. Diese Ausdrücke verbessern frühere graphische und empirische Korrelationen sowohl hinsichtlich der Genauigkeit als auch der Praktikabilität.

КОРРЕЛЯЦИОННЫЕ УРАВНЕНИЯ ЛАМИНАРНОЙ И ТУРБУЛЕНТНОЙ СВОБОДНОЙ КОНВЕКЦИИ ОТ ГОРИЗОНТАЛЬНОГО ЦИЛИНДРА

Аннотация — С помощью модели Черчилля и Узаги получено простое эмпирическое выражение для среднего по цилиндру значения числа при всех значениях *Ra* и *Pr*. Данное выражение применимо к случаю однородного нагрева, а также к случаям однородной температуры стенки и массообмена, и одновременного тепло- и массообмена. Для ограниченных условий получены более простые выражения. Эти выражения сделали имеющиеся графические и эмпирические соотношения более точными и удобными.